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Abstract. We describe the Wadge hierarchy of the ω-languages recognized by deterministic Petri nets. This is an extension of the celebrated Wagner hierarchy which turned out to be the Wadge hierarchy of the ω-regular languages. Petri nets are an improvement of automata. They may be defined as partially blind multi-counter automata. We show that the whole hierarchy has height ω^{ω^2}, and give a description of the restrictions of this hierarchy to every fixed number of partially blind counters.

1 Introduction

The languages of infinite words – also called ω-languages – that are accepted by finite automata were first studied by Büchi in order to prove the decidability of the monadic second order theory of one successor over the integers. Since then, the regular ω-languages have been intensively studied, mostly for applications to specification and verification of non-terminating systems. See [29, 40, 41] for many results and references. Following this trend, the acceptance of infinite words by other types of finite machines, such as pushdown automata, multi-counter automata, Petri nets, or even Turing machines, were later considered [4, 9, 20, 32, 40].

Since the set of infinite words over a finite alphabet becomes a topological space once equipped with the Cantor topology, a way to study the complexity of the languages of infinite words accepted by finite machines is to study their topological complexity. This consists in providing their precise localization inside the projective hierarchy, the Borel hierarchy, or even the Wadge hierarchy (a great refinement of the Borel hierarchy). This work was conducted through [9, 25, 33, 35, 37, 38, 39, 40, 41].

It is well known that every ω-language accepted by a deterministic Büchi automaton is a $\mathbf{\Pi}^0_2$-set, and that an ω-language accepted by a non-deterministic Büchi (or Muller) automaton is a $\mathbf{\Delta}^0_3$-set. The Borel hierarchy of regular ω-languages is then determined. Moreover, Landweber proved that one can effectively determine the Borel complexity of a regular ω-language accepted by a given Muller or Büchi automaton, see [24, 29, 40, 41]. Elaborating on this result, Klaus Wagner completely described the Wadge hierarchy of the ω-regular languages [44]. It is nowadays called the Wagner hierarchy, and its length is the ordinal ω^ω. Wagner gave an automaton-like characterization of this hierarchy, based on the notions of chain and superchain, together with an algorithm to compute the Wadge (Wagner) degree of any given ω-regular language. Later, Wilke and Yoo proved that the Wadge degree of an ω-regular language may be computed in polynomial time [45]. This hierarchy was thouroughly studied by Carton and Perrin in [2, 3], and by Victor Selivanov in [31, 34].

Since there are various classes of finite machines recognizing ω-languages, each of them yields a countable sub-hierarchy of the Wadge hierarchy. Since the 1980's it has been an endeavor to describe these sub-hierarchies. It started with the work of Klaus Wagner on the ω-regular languages – although Wagner was unaware at the time of the connections between the Wadge hierarchy and his own work. The Wadge hierarchy of deterministic context-free ω-languages was determined, together with its length: $\omega^{(\omega^2)}$ [6, 7]. The problem whether this hierarchy is decidable remains open. The Wadge hierarchy induced by the subclass of deterministic one blind counter automata was determined in an effective way [11], and other partial decidability results were obtained [12]. It was then proved that the Wadge hierarchy of context-free ω-languages is the same as the one of effective analytic sets[3] [15, 20]. Intriguingly, the only Wadge class for which one can decide whether a given context-free ω-language belongs to or not, is the rudimentary singleton $\{\emptyset\}$, see [12, 13, 14]. In particular, one cannot decide whether a non-deterministic pushdown automaton is universal or not. This latter decision problem is actually Π^1_2-complete, hence located at the second level of the analytical hierarchy and "highly undecidable", [18]. Moreover the second author proved that the topological complexity of some context-free ω-languages may be subject to change from one model of set theory to another [17]. (Similar results hold for ω-languages accepted by 2-tape Büchi automata [16, 17].) Finally, the Wadge hierarchy of ω-languages of deterministic Turing machines was determined by Victor Selivanov, [32].

Petri nets are among the many accepting devices that are more powerful than finite automata in that they recognize more ω-languages that finite automata. They apply to the description of distributed systems. A Petri net is a directed bipartite graph, in which the nodes represent transitions and places. The distributions of tokens over the places define the configurations of the net. Petri nets work as an improvement of automata, since they may be defined as *partially blind multicounter automata* [21]. Petri nets have been extensively examined, particularly in concurrency theory (see for instance [10, 30]). The infinite behavior of Petri nets was first studied by Valk [42], and the one of deterministic Petri nets, by Carstensen [1].

In this paper, we first consider deterministic blind multicounter automata (corresponding to deterministic Petri nets) and the ω-languages that they accept when they are equipped with a Muller acceptance condition. This forms the class of deterministic Petri net ω-languages denoted $\mathcal{L}^3_{\omega dt}$ in [1].

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[3] The class of all effective analytic sets (denoted Σ^1_1) is the class of all the ω-languages recognized by (non-deterministic) Turing machines.

We describe the Wadge hierarchy of the ω-languages recognized by deterministic Petri nets. This is an extension of the celebrated Wagner hierarchy of the ω-regular languages. We show that the whole hierarchy has height ω^{ω^2}, and give a description of the restrictions of this hierarchy to some fixed number of partially blind counters.

2 Recalls on ω-languages, automata and Petri nets

We assume the reader to be familiar with the theories of formal languages and ω-regular languages (see [22, 29, 41]).
Through along the paper, we assume Σ to be any finite set, called the alphabet. A finite word (string) over Σ is any sequence of the form $u = a_1 \ldots a_k$, where $k \in \mathbb{N}$ and $a_i \in \Sigma$ holds for each $i \leq k$. Notice that when $k = 0$, u is the empty word denoted by ε. We denote by $|u|$ the length of the word u (here $|u| = k$). We write $u(i) = a_i$ and $u[i] = u(1) \ldots u(i)$ for $i \leq k$ and $u[0] = \varepsilon$. The set of all finite words over Σ is denoted Σ^*.
An infinite word over Σ is some sequence of the form $x = a_1 a_2 \ldots a_n \ldots$ where $a_i \in \Sigma$ holds for all non-zero integers i. These infinite words are called ω-words for their length corresponds to ω: the first infinite ordinal. An infinite word x over Σ can be viewed as a mapping $x : \mathbb{N} \longrightarrow \Sigma$, so we write $x = x(1)x(2)\ldots$ and $x[n] = x(1)x(2)\ldots x(n)$ for its prefix of length n[4]. We write Σ^ω for the set of all ω-words over the alphabet Σ, so that an ω-language over the alphabet Σ is nothing but a subset of Σ^ω.
As usual, the concatenation of two finite words u and v is denoted uv. It naturally extends to the concatenation of a finite word u and an ω-word x to give the ω-words $y = ux$ defined by: $y(k) = u(k)$ if $k \leq |u|$, and $y(k) = x(k-|u|)$ if $k > |u|$. Given any finite word u, and any finite or infinite word x, u is a prefix of x (denoted $u \sqsubseteq x$) if $u(i) = x(i)$ holds for every non-zero integer $i \leq |u|$. Finally, for $V \subseteq \Sigma^*$, $V^\omega = \{\sigma = u_1 \ldots u_n \ldots \in \Sigma^\omega \mid u_i \in V, \forall i \geq 1\}$.

A *finite state machine (FSM)* is a quadruple $M = (Q, \Sigma, \delta, q_0)$, where Q is a finite set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state and δ is a mapping from $Q \times \Sigma$ into 2^Q . It is *deterministic (DFSM)* if $\delta : Q \times \Sigma \longrightarrow Q$.

Given an infinite word x, the infinite sequence of states $\rho = q_1 q_2 q_3 \ldots$ is called an (infinite) run of M on x starting in state p, if both $q_1 = p$ and $q_{i+1} \in \delta(q_i, a_i)$ $(\forall i \geq 1)$ hold. In case p is the initial state of M $(p = q_0)$, then ρ is simply called an infinite run of M on x.
We denote by $In(\rho) = \{q \in Q \mid \forall m\, \exists n > m\ q_n = q\}$ the set of states that appear infinitely often in ρ.

Equipped with an acceptance condition F, a finite state machine becomes a finite state automaton $M = (Q, \Sigma, \delta, q_0, F)$. It is a *Büchi automaton (BA)* when $F \subseteq Q$, and a *Muller automaton (MA)* when $F \subseteq 2^Q$. A Büchi automaton

[4] note that the enumeration $x = x(1)x(2)\ldots$ does not start at 0 so that we recover the empty word as $x[0]$.

(respectively a Muller automaton) accepts x if for some infinite run of M on x, $In(\rho) \cap F$ is not empty (respectively $In(\rho) \in F$ holds). The ω-language accepted by an automaton is the set of all the infinite words it accepts. The classical result of R. Mc Naughton [28] establishes that non-deterministic Büchi automata, and both deterministic and non-deterministic Muller automata recognize the exact same ω-languages known as the ω-regular languages[5].

A partially blind multicounter automaton is a finite automaton equipped with a finite number (k) of partially blind counters. The content of any such counter is a non-negative integer. A counter is said to be partially blind when the multicounter automaton cannot test whether the content of the counter is zero. This means that if a transition of the machine is enabled when the content of a counter is zero then the same transition is also enabled when the content of the same counter is a non-zero integer. In order to get a partially blind multicounter automaton – simply called a *blind multicounter automaton* – which accepts the same language as a given Petri net, one can distinguish between the places of a Petri net by dividing them into the bounded ones (the number of tokens in such a place at any time is uniformly bounded) and the unbounded ones. Then each unbounded place may be seen as a blind counter, and the tokens in the bounded places determine the state of the blind multicounter automaton. The transitions of the Petri net may then be seen as the finite control of the blind multicounter automaton and the labels of these transitions are then the input symbols.
Contrary to what happens with non-deterministic Petri nets, allowing ε-transitions does not increase the expressive power of deterministic Petri nets which read ω-words [1]. For this reason, we restrict ourselves to the sole real time – *i.e.*, ε-transition free – blind multicounter automata. Also, without loss of generality we may assume that every transition, for every counter, either increases or decreases its content by 1 or leaves it untouched.

Definition 1. *For k any non-zero integer, A (real time) deterministic k-blind-counter machine (k-BCM) is of the form $\mathcal{M} = (Q, \Sigma, \delta, q_0)$ where Q is a finite set of states, Σ is a finite input alphabet, $q_0 \in Q$ is the initial state, and the transition relation δ is a partial mapping from $Q \times \Sigma \times \{0,1\}^k$ into $Q \times \{0,1,-1\}^k$.*

If the machine $\mathcal{M}$ is in state q, and for each i, $c_i \in \mathbb{N}$ is the content of the counter $\mathcal{C}_i$, then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

Given any $a \in \Sigma$, $q, q' \in Q$, and $(c_1, \ldots, c_k) \in \mathbb{N}^k$, if both $\delta(q, a, i_1, \ldots, i_k) = (q', j_1, \ldots, j_k)$, and $j_l \in E = \{l \in \{1, \ldots, k\} \mid c_l = 0\} \Rightarrow j_l \in \{0,1\}$ hold, then we write $a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k)$. Thus the transition relation must verify: if $\delta(q, a, i_1, \ldots, i_k) = (q', j_1, \ldots, j_k)$, and $i_m = 0$ holds for some $m \in \{1, \ldots, k\}$, then we must have $j_m = 0$ or $j_m = 1$ (but $j_m = -1$ is prohibited).

[5] The class of all the ω-regular languages is also characterized as the "ω-Kleene closure" of the class REG of all the (finitary) regular languages. Where given any class of finitary languages $\mathcal{L}$, the ω-Kleene closure of $\mathcal{L}$ is the class of ω-languages $\{\bigcup_{1 \leq i \leq n} U_i.V_i^{\omega} \mid U_i, V_i \in \mathcal{L}\}$.

Moreover the k counters of $\mathcal{M}$ are blind, i.e., *if* $\delta(q, a, i_1, \ldots, i_k) = (q', j_1, \ldots, j_k)$ *holds, and* $i_m = 0$ *for* $m \in E \subseteq \{1, \ldots, k\}$, *then* $\delta(q, a, i'_1, \ldots, i'_k) = (q', j_1, \ldots, j_k)$ *holds also whenever* $i_m = i'_m$ *for* $m \notin E$, *and* $i'_m = 0$ *or* $i'_m = 1$ *for* $m \in E$.

For any finite word $u = a_1 a_2 \ldots a_n$ *over* Σ, *a sequence of configurations* $\rho = (q_i, c^i_1, \ldots c^i_k)_{1 \leq i \leq n+1}$ *is a run of* $\mathcal{M}$ *on* u, *starting in configuration* $(p, c_1, \ldots, c_k)$ *iff* $(q_1, c^1_1, \ldots, c^1_k) = (p, c_1, \ldots, c_k)$, *and* $a_i : (q_i, c^i_1, \ldots c^i_k) \mapsto_{\mathcal{M}} (q_{i+1}, c^{i+1}_1, \ldots c^{i+1}_k)$ *(all* $1 \leq i \leq n$*). This notion extends naturally to infinite words: for* $x = a_1 a_2 \ldots a_n \ldots$ *any* ω*-word over* Σ, *an* ω*-sequence of configurations* $(q_i, c^i_1, \ldots c^i_k)_{i \geq 1}$ *is called a complete run of* $\mathcal{M}$ *on* x, *starting in configuration* $(p, c_1, \ldots, c_k)$ *iff* $(q_1, c^1_1, \ldots c^1_k) = (p, c_1, \ldots, c_k)$, *and* $a_i : (q_i, c^i_1, \ldots c^i_k) \mapsto_{\mathcal{M}} (q_{i+1}, c^{i+1}_1, \ldots c^{i+1}_k)$ *(for all* $1 \leq i$*).*
A complete run ρ *of* $\mathcal{M}$ *on* x, *starting in configuration* $(q_0, 0, \ldots, 0)$, *is simply called "a run of* $\mathcal{M}$ *on* x*".*

Definition 2. *A Büchi (resp. Muller) deterministic* k*-blind-counter automaton is some* k*-BCM* $\mathcal{M}' = (Q, \Sigma, \delta, q_0)$, *equipped with an acceptance condition* F: $\mathcal{M} = (Q, \Sigma, \delta, q_0, F)$. *It is a Büchi (resp. Muller*[6]*)* k*-blind-counter automaton when* $F \subseteq Q$ *(resp.* $F \subseteq 2^Q$*), and it accepts* x *if the infinite run of* $\mathcal{M}'$ *on* x *verifies* $In(\rho) \cap F \neq \emptyset$ *(respectively* $In(\rho) \in F$*).*
We write $L(\mathcal{M})$ *for the* ω*-language accepted by* $\mathcal{M}$, *and* $\mathbf{BC(k)}$ *for the class of* ω*-languages accepted by Muller deterministic* k*-blind-counter automata.*

3 Borel and Wadge hierarchies

We assume the reader to be familiar with basic notions of topology that may be found in [23, 25, 27], and of ordinals (in particular the operations of multiplication and exponentiation) that may be found in [36].
For any given finite alphabet X – that contains at least two letters – we consider X^ω as the topological space equipped with the Cantor topology[7]. The open sets of X^ω are those of the form WX^ω, for some $W \subseteq X^*$. The closed sets are the complements of the open sets. The class that contains both the open sets and the closed sets, and is closed under countable union and intersection is the class of Borel sets. It is nicely set up in a hierarchy but counting how many times these latter operations are needed.
This defines the Borel Hierarchy: $\mathbf{\Sigma}^0_1$ is the class of open sets , and $\mathbf{\Pi}^0_1$ is the class of closed sets. For any non-zero integer n, $\mathbf{\Sigma}^0_{n+1}$ is the class of countable unions of sets inside $\mathbf{\Pi}^0_n$, while $\mathbf{\Pi}^0_{n+1}$ is the class of countable intersections of sets inside $\mathbf{\Sigma}^0_n$. More generally, for any non-zero countable ordinal α, $\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of sets in $\cup_{\gamma<\alpha}\mathbf{\Pi}^0_\gamma$, and $\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of sets in $\cup_{\gamma<\alpha}\mathbf{\Sigma}^0_\gamma$.

[6] The Muller acceptance condition was denoted 3-acceptance in [24, 1], and $(inf, =)$ in [40].
[7] The product topology of the discrete topology on X.

The Borel rank of a subset A of X^ω is the least ordinal $\alpha \geq 1$ such that A belongs to $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha$. By ways of continuous pre-image, the Borel hierarchy turns into the refined Wadge Hierarchy.

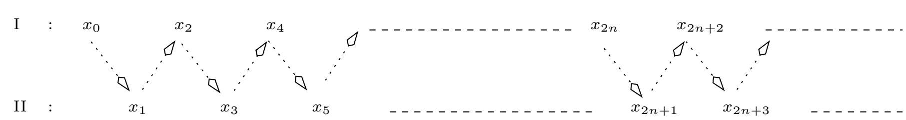
Definition 3 ($\leq_w, \equiv_w, <_w$). *We let X, Y be two finite alphabets, and $A \subseteq X^\omega, B \subseteq Y^\omega$, A is said Wadge reducible to B (denoted $A \leq_W B$) iff there exists some continuous function $f : X^\omega \longrightarrow Y^\omega$ that satisfies $\forall x \in X^\omega \;\; (x \in A \Leftrightarrow f(x) \in B)$.*

We write $A \equiv_w B$ for $A \leq_w B \leq_w A$, and $A <_w B$ for $A \leq_w B \not\leq_w A$. A set $A \subseteq X^\omega$ is *self dual* if $A \equiv_w X^\omega \smallsetminus A$ (denoted $A^{\complement}$) is verified. It is *non-self dual* otherwise [8].

It is easy to verify that the relation $\leq_w$ is both reflexive and transitive, and that $\equiv_w$ is an equivalence relation. Given any set A, the class of all its continuous pre-images forms a topological[9] class $\mathbf{\Gamma}$ called a Wadge class. A set is $\mathbf{\Gamma}$-*complete* if it both belongs to $\mathbf{\Gamma}$, and (Wadge) reduces every element in it[10]. It turns out that $\mathbf{\Sigma}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha$) is a Wadge class and any set in $\mathbf{\Sigma}^0_\alpha \smallsetminus \mathbf{\Pi}^0_\alpha$ (resp. $\mathbf{\Pi}^0_\alpha \smallsetminus \mathbf{\Sigma}^0_\alpha$) is $\mathbf{\Sigma}^0_\alpha$-complete (resp. $\mathbf{\Pi}^0_\alpha$-complete). Both $\mathbf{\Sigma}^0_n$-complete and $\mathbf{\Pi}^0_n$-complete sets (any $0 < n < \omega$) are examined in [38].

Wadge reducibility participates in game theory for continuous functions may be regarded as strategies for a player in a two-player game of perfect information and infinite length:

Definition 4. *Given any mapping $f : X^\omega \longrightarrow Y^\omega$, the game $\mathbf{G}(f)$ is the two-player game where players take turn picking letters in X for I and Y for II, player I starting the game, and player II being allowed in addition to pass her turn, while player I is not.*



After ω-many moves, player I and player II have respectively constructed $x \in X^\omega$ and $y \in Y^* \cup Y^\omega$. Player II wins the game if $y = f(x)$, otherwise player I wins.

So, in the game $\mathbf{G}(f)$, a strategy for player I is a mapping $\sigma : (Y \cup \{s\})^\star \longrightarrow X$, where s is a new letter not in Y that stands for II's moves when she passes her turn[11]. A strategy for player II is a mapping $f : X^+ \longrightarrow Y \cup \{s\}$. A strategy is called winning if it ensures a win whatever the opponent does.

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[8] Non-self dual sets are precisely those that verify $A \not\leq_w A^{\complement}$.

[9] A topological class is a class that is closed under continuous pre-images.

[10] It follows that two sets are complete for the same topological class iff they are Wadge equivalent.

[11] "s" stands for "skips".

This game was designed to characterize the continuous functions. Wadge found out that given $f : X^\omega \longrightarrow Y^\omega$, f is continuous $\iff$ *II* has a winning strategy in $\mathbf{G}(f)$. This is an easy exercise (see [23, 27]).

Definition 5. *For $A \subseteq X^\omega$ and $B \subseteq Y^\omega$, the Wadge game $\mathbf{W}(A, B)$ is the same as $\mathbf{G}(f)$, except that II wins iff $y \in Y^\omega$ and $(x \in A \iff y \in B)$ hold.*[12]

In 1975, Martin proved Borel determinacy [23, 26], whose consequence is that for every Wadge game $\mathbf{W}(A, B)$, either player I or II has a winning strategy as long as both A and B are Borel. As immediate consequences, Wadge obtained that for any Borel $A, B \subseteq X^\omega$, there are no three $\leq_w$-incomparable Borel sets. Moreover, if $A \not\leq_w B$ and $B \not\leq_w A$, then $A \equiv_w B^{\complement}$. Later on, Martin and Monk proved that there is no sequence $(A_i)_{i\in\omega}$ of Borel subsets of X^ω such that $A_0 >_w A_1 >_w A_2 >_w \ldots A_n >_w A_{n+1} >_w \ldots$ holds [23, 43]. We recall that a set S is well ordered by the binary relation $<$ on S iff $<$ is a linear order on S such that there is no strictly infinite $<$-decreasing sequence of elements from S.
It follows that up to complementation and $\equiv_w$, the class of Borel subsets of X^ω, is well-ordered by $<_w$. Therefore, there is a unique ordinal $|WH|$ isomorphic to this well-ordering, together with a mapping d_W^0 from the Borel subsets of X^ω onto $|WH|$, such that for all Borel subsets A, B: $d_W^0 A < d_W^0 B \Leftrightarrow A <_w B$, and $d_W^0 A = d_W^0 B \Leftrightarrow (A \equiv_w B$ or $A \equiv_w B^{\complement})$.
This well-ordering restricted to the Borel sets of finite ranks[13] has length the first ordinal that is a fixpoint of the operation $\alpha \longrightarrow {\omega_1}^\alpha$ [5, 43], where ω_1 is the first uncountable ordinal.

In order to study the Wadge hierarchy of the class $\mathbf{BC(k)}$ of ω-languages accepted by Muller deterministic k-blind-counter automata, we concentrate on the non-self dual sets as in [5], and slightly modify the definition of the Wadge degree. For $A \subseteq X^\omega$, such that $A >_w \emptyset$, we set $d_w(\emptyset) = d_w(\emptyset^{\complement}) = 1$, $d_w(A) = sup\{d_w(B) + 1 \mid B$ non-self dual and $B <_W A\}$.

Every ω-language which is accepted by a deterministic Petri net – more generally by a deterministic $\mathbf{X}$-automaton in the sense of [9] or by a deterministic Turing machine – is a boolean combination of $\mathbf{\Sigma_2^0}$-sets thus its Wadge degree inside the whole Wadge hierarchy of Borel sets is located below ω_1^ω. Moreover, every ordinal $0 < \alpha < \omega_1^\omega$ admits a unique Cantor normal form of base ω_1 [36], *i.e.*, it can be written as $\alpha = \omega_1^{n_j}.\delta_j + \omega_1^{n_{j-1}}.\delta_{j-1} + \cdots + \omega_1^{n_1}.\delta_1$ where $0 < j < \omega$, $0 \leq n_1 < \ldots < n_j < \omega$, and $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are non-zero countable ordinals.

From Wagner's study, such an ordinal is the Wadge degree of an ω-regular language iff $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are all integers. It is also known that such an ordinal

[12] One sees immediately that a winning strategy for *II* in $\mathbf{W}(A, B)$ yields a continuous mapping $f : X^\omega \longrightarrow Y^\omega$ that guaranties that $A \leq_w B$ holds, whereas any continuous function f that witnesses the reduction relation $A \leq_w B$ gives rise to some winning strategy for *II* in $\mathbf{G}(f)$ which is also winning for *II* in $\mathbf{W}(A, B)$. This shows that for $A \subseteq X^\omega$ and $B \subseteq Y^\omega$, $A \leq_w B \iff$ *II has a winning strategy in* $\mathbf{W}(A, B)$.

[13] The Borel sets of finite ranks are those in $\bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}_n^0 = \bigcup_{n\in\mathbb{N}} \mathbf{\Pi}_n^0$.

is the Wadge degree of a deterministic context-free ω-language if and only if these multiplicative coefficients are all below ω^ω [6]. We add to this picture the following results that exhibits the Wadge hierarchy of $\mathbf{BC(k)}$:

1. for every non-null ordinal α whose Cantor normal form of base ω_1 is

$$\alpha = \omega_1^{n_j}.\delta_j + \omega_1^{n_{j-1}}.\delta_{j-1} + \cdots + \omega_1^{n_1}.\delta_1$$

   where, for some integer $k \geq 1$, $\delta_1, \ldots, \delta_j$ are (non-null) ordinals $< \omega^{k+1}$, there exists some ω-language $L \in \mathbf{BC(k)}$ whose Wadge degree is α.
2. Non-self dual ω-languages in $\mathbf{BC(k)}$ have Wadge degrees of the above form.

Next section is dedicated to operations that will be needed in the proof.

4 Operations over sets of ω-words

4.1 The sum

Definition 6. *For $\{X_+, X_-\}$ a partition in non-empty sets of $X_B \setminus X_A$ with $X_A \subseteq X_B$, $A \subseteq X_A^\omega$, and $B \subseteq X_B^\omega$, $B + A = A \ \cup \ X_A^* X_+ B \ \cup \ X_A^* X_- B^{\complement}$.*

A player in charge of $B + A$ in a Wadge game is like a player who begins the play in charge of A, and at any moment may also decide to start anew but being in charge this time of either B or of $B^{\complement}$ [14].

Proposition 7 (Wadge). *For non-self dual Borel sets A and B,*

$$d_w(B + A) = d_w(B) + d_w(A).$$

Notice that for any non-self dual Borel sets A, B, C, we have both $A+(B+C) \equiv_w (A+B)+C$, and $(B+A)^{\complement} \equiv_w B + A^{\complement}$. Although the class $\mathbf{BC(k)}$ is not closed under complementation, and $B + A$ was defined as $A \ \cup \ X_A^* X_+ B \ \cup \ X_A^* X_- B^{\complement}$, we may however use of the formulation $B + A \in \mathbf{BC(k)}$ for $A, B \in \mathbf{BC(k)}$ if some $C \in \mathbf{BC(k)}$ verifies $C \equiv_w B^{\complement}$.

4.2 The countable multiplication

We first need to define the supremum of a countable family of sets.

[14] The first letter in $X_B \setminus X_A$ that is played decides the choice of B or $B^{\complement}$. Notice that given any finite alphabets X, Y which contain at least two letters, and any $B \subseteq X^\omega$, there exists $B' \subseteq Y^\omega$ such that $B \equiv_w B'$. Moreover, if for some integer $k \geq 0$ we have $B \in \mathbf{BC(k)}$, then B' can be taken in $\mathbf{BC(k)}$. So that we may write $B + A$ whatever space B is a subset of, simply meaning $B' + A$ where B' is any set that satisfies both $B' \equiv_w B$ and $B' \subseteq X^\omega$ for some X that contains the alphabet from which A is taken from, and strictly extends it with at least two new letters.

Definition 8. *For any bijection $f : \mathbb{N} \longrightarrow I$, any family $(A_i)_{i \in I}$ of non-self dual Borel subsets of X^ω, we fix some letter $e \in X$ to define*

$$\sup_{i \in I} A_i = \bigcup_{n \in \mathbb{N}} (X \setminus \{e\})^n e A_{f(n)}.$$

Proposition 9. *(See [5, 6].) For $(A_i)_{i \in I}$ any countable family of non-self dual Borel subsets of X^ω such that $\forall i \in I \quad \exists j \in I \quad A_i <_w A_j$, then*

1. *$\sup_{i \in I} A_i$ is a non-self dual Borel subset of X^ω, and*
2. *$d_w(\sup_{i \in I} A_i) = \sup\{d_w(A_i) \mid i \in I\}$.*

By combining sum and supremum, we get multiplication by countable ordinals.

Definition 10. *For $A \subseteq X^\omega$, and $0 < \alpha < \omega_1$, $A \bullet \alpha$ is inductively defined by $A \bullet 1 = A$, $A \bullet (\nu + 1) = (A \bullet \nu) + A$, and $A \bullet \beta = \sup_{\delta \in \beta} A \bullet \delta$, for β limit.*

By Propositions 7 and 9, this operation verifies the following.

Proposition 11. *Let $A \subseteq X^\omega$ be some non-self dual Borel set, and $0 < \alpha < \omega_1$,*

$$d_w(A \bullet \alpha) = d_w(A) \cdot \alpha.$$

For a player in charge of $A \bullet \alpha$ in a Wadge game, everything goes as if (s)he could switch again and again between being in charge of A or $A^\complement$ – starting anew every time (s)he does so – but restrained from doing so infinitely often by having to construct a decreasing sequence of ordinals $< \alpha$ on the side every time (s)he switches.

4.3 The multiplication by ω_1

Definition 12. *For $A \subseteq X^\omega$, and $a, b \notin X$ two different letters, $Y = X \cup \{a, b\}$, $A \bullet \omega_1 \subseteq (X \cup \{a, b\})^\omega$ is defined[15] by $A \bullet \omega_1 = A \ \cup \ Y^* a A \ \cup \ Y^* b A^\complement$.*

Inside a Wadge game, a player in charge of $A \bullet \omega_1$ may switch indefinitely between being in charge of A or its complement, deleting what (s)he has already played each time.

Proposition 13. *(See [5].) For any non-self dual Borel $A \subseteq X^\omega$, $A \bullet \omega_1$ is non-self dual Borel, and $d_w(A \bullet \omega_1) = d_w(A) \cdot \omega_1$.*

The following property will be very useful.

Proposition 14. *If $A \subseteq X^\omega$ is regular, then $A \bullet \omega_1$ is also regular.*

Proof. Immediate from the closure of the class REG_ω under finite union, complementation, and left concatenation by finitary regular languages [7]. $\square$

[15] This operation was denoted $A \longrightarrow A\hat{}\infty$ in [7], and $A \longrightarrow A^\natural$ in [6].

4.4 Canonical non-self dual sets

The empty set, considered as an ω-language over a finite alphabet is a Borel set of Wadge degree 1, *i.e.*, $d_w(\emptyset) = 1$. It is a non-self dual set and its complement has the same Wadge degree[16]. On the basis of the emptyset or its complement, the operations defined above provide non-self dual Borel sets for every Wadge degree $< \omega_1^\omega$. For notational purposes, given any $A \subseteq X^\omega$ we define $A \bullet \omega_1^n$ by induction on $n \in \mathbb{N}$ by: $A \bullet \omega_1^0 = A$, and $A \bullet \omega_1^{n+1} = (A \bullet \omega_1^n) \bullet \omega_1$.
Clearly, by Proposition 13, $d_w(A \bullet \omega_1^n) = d_w(A) \cdot \omega_1^n$ holds for every non-self dual Borel $A \subseteq X^\omega$. It follows that the ω-language $\emptyset \bullet \omega_1^n$ is a non-self dual Borel set whose Wadge degree is precisely ω_1^n.
Every non-null ordinal $\alpha < \omega_1^\omega$ admits a unique Cantor normal form of base ω_1:

$$\alpha = \omega_1^{n_j} \cdot \delta_j + \omega_1^{n_{j-1}} \cdot \delta_{j-1} + \cdots + \omega_1^{n_1} \cdot \delta_1.$$

where $\omega > j > 0$, $\omega > n_j > n_{j-1} > \ldots > n_1 \geq 0$, and $\delta_j, \delta_{j-1}, \ldots, \delta_1$ are non-zero countable ordinals [36].
As in [5, 6], we set $\Omega(\alpha) := (\emptyset \bullet \omega_1^{n_j}) \bullet \delta_j + (\emptyset \bullet \omega_1^{n_{j-1}}) \bullet \delta_{j-1} + \cdots + (\emptyset \bullet \omega_1^{n_1}) \bullet \delta_1$.
By Propositions 7, 11, and 13 $d_w(\Omega(\alpha)) = \alpha$ holds.

5 A hierarchy of $BC(k)$

From now on, we restrain ourselves to the sole ordinals $\alpha < \omega_1^\omega$ whose Cantor normal form of base ω_1 contains only multiplicative coefficients strictly below ω^{k+1}, and we construct for every such α some Muller deterministic k-blind-counter automata $\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^-$ such that both $L(\mathcal{M}_\alpha) \equiv_w \Omega(\alpha)$ and $L(\mathcal{M}_\alpha^-) \equiv_w \Omega(\alpha)^\complement$ hold.

To start with, notice that for every integer n since $\emptyset \bullet \omega^n \in REG_\omega$ is verified, there exist deterministic Muller automata $\mathcal{O}_n = (Q_n, X_n, \delta_n, q_n^0, \mathcal{F}_n)$, where $\mathcal{F}_n \subseteq 2^{Q_n}$ is the collection of designated state sets, such that $L(\mathcal{O}_n) = \emptyset \bullet \omega^n$. We prove the following results:

Proposition 15. *For any ω-regular language A, any integer $j \geq 1$ there exist ω-languages $B, C \in \mathbf{BC(j)}$ such that $B \equiv_w (A \bullet \omega^j)$ and $C \equiv_w (A \bullet \omega^j)^\complement$.*

A careful generalization of the ideas of the proofs of Proposition 15 leads to:

Proposition 16. *For any ω-regular A, integer k, and ordinal $\omega^k \leq \alpha < \omega^{k+1}$, there exist $B, C \in \mathbf{BC(k)}$ such that both $B \equiv_w (A \bullet \alpha)$ and $C \equiv_w (A \bullet \alpha)^\complement$ hold.*

Theorem 17. *Let $\alpha < \omega_1^\omega$ be any ordinal of the form*

$$\alpha = \omega_1^{n_j} \cdot \delta_j + \omega_1^{n_{j-1}} \cdot \delta_{j-1} + \cdots + \omega_1^{n_0} \cdot \delta_0$$

where $\omega > j \geq 0$, $\omega > n_j > n_{j-1} > \ldots > n_0 \geq 0$, and $\omega^\omega > \delta_j, \delta_{j-1}, \ldots, \delta_0 > 0$. Let k be the least integer such that $\forall i \leq j \quad \delta_i < \omega^{k+1}$. Then there exist ω-languages $B, C \in \mathbf{BC(k)}$ such that $B \equiv_w \Omega(\alpha)$ and $C \equiv_w \Omega(\alpha)^\complement$.

We recall that $\Omega(\alpha) := (\emptyset \bullet \omega_1^{n_j}) \bullet \delta_j + (\emptyset \bullet \omega_1^{n_{j-1}}) \bullet \delta_{j-1} + \cdots + (\emptyset \bullet \omega_1^{n_0}) \bullet \delta_0$.

[16] *i.e.*, $d_w(\emptyset) = d_w(X^\omega) = 1$.

6 Localisation of $BC(k)$

This section is dedicated to proving that there is no other Wadge class generated by some non-self dual ω-language in $BC(k)$ than the ones described in Theorem 17. Prior to this we need a technical result about the Wadge hierarchy together with a few others on ordinal combinatorics, and notations.

For some $A \subseteq X^\omega$ and $u \in X^*$, we write $u^{-1}A$ for the set $\{x \in X^\omega \mid ux \in A\}$. We say that A is *initializable* if player *II* has a w.s. in the Wadge game $\mathbf{W}(A, A)$ even though she is restricted to positions $u \in X^*$ that verify $u^{-1}A \equiv_w A$.

Lemma 18. *For $A \subseteq X^\omega$ any initializable set, $B \subseteq Y^\omega$, and δ, θ any countable ordinals,*

$$A \bullet (\theta + 1) \leq_w B \leq_w A \bullet \delta \quad \Longrightarrow \quad \exists u \in Y^* \begin{cases} u^{-1}B \equiv_w A \bullet (\theta + 1) \\ \quad\quad or \\ u^{-1}B \equiv_w (A \bullet (\theta + 1))^{\complement}. \end{cases}$$

Lemma 19. *We let $B \subseteq Y^\omega$, $A \subseteq X^\omega$ be any initializable set, and δ, θ be any countable ordinals. We consider any set of the form*

$$C = A \bullet \omega_1^n \bullet \nu_n + \cdots + A \bullet \omega_1^{n-1} \bullet \nu_{n-1} + \cdots + A \bullet \omega_1 \bullet \nu_1$$

for any non-zero integer n, and countable coefficients $\nu_n, \nu_{n-1}, \ldots, \nu_1$ with at least one of them being non-null.

$$C + A \bullet (\theta + 1) \leq_w B \leq_w C + A \bullet \delta \quad \Longrightarrow \quad \exists u \in Y^* \begin{cases} u^{-1}B \equiv_w C + A \bullet (\theta + 1) \\ \quad\quad or \\ u^{-1}B \equiv_w (C + A \bullet (\theta + 1))^{\complement}. \end{cases}$$

We recall that for any set of ordinals $\mathcal{O}$, its order type – denoted $ot(\mathcal{O})$ – is the unique ordinal that is isomorphic to $\mathcal{O}$ ordered by membership.

Definition 20. *The function $\mathcal{H} : \omega^\omega \times \omega^\omega \longrightarrow On$ is defined by*

$$\mathcal{H}(\alpha, \beta) \ = \ \omega^k \cdot (l_k + m_k) + \omega^{k-1} \cdot (l_{k-1} + m_{k-1}) + \cdots + \omega^0 \cdot (l_0 + m_0).$$

Where (a variation of the) the Cantor normal form of base ω of α (resp. β) is $\alpha = \omega^k \cdot l_k + \omega^{k-1} \cdot l_{k-1} + \cdots + \omega^0 \cdot l_0$, $\beta = \omega^k \cdot m_k + \omega^{k-1} \cdot m_{k-1} + \cdots + \omega^0 \cdot m_0$, with $l_k, m_k, l_{k-1}, m_{k-1}, \ldots, l_0, m_0 \in \mathbb{N}$. (Some of these integers may be null[17].)

Lemma 21. *Let $\mathcal{H} : \omega^\omega \times \omega^\omega \longrightarrow On$, $0 < \alpha', \alpha, \beta' \beta < \omega^\omega$ with $\alpha' \leq \alpha$, $\beta' \leq \beta$ but either $\alpha' < \alpha$ or $\beta' < \beta$, then $\mathcal{H}(\alpha', \beta') < \mathcal{H}(\alpha, \beta)$.*

We make use of the mapping $\mathcal{H}$ to prove the following combinatorial result.

Lemma 22. *Let α, β, γ be non-null ordinals with $\alpha, \beta < \omega^\omega$, and $f : \gamma \longrightarrow$* $\{0, 1\}$. If both $\alpha = ot(f^{-1}[0])$ and $\beta = ot(f^{-1}[1])$ hold, then $\gamma \leq \mathcal{H}(\alpha, \beta)$.

[17] In particular, $l_k, l_{k-1}, \ldots$ $m_k, m_{k-1}, \ldots$ might be null, but since $\alpha, \beta > 0$ holds, at least one of the l_i's, and one of the m_i's are different from zero.

Corollary 23. *Let k, n be non-null integers, γ be any ordinal, $0 \leq \alpha_0, \ldots, \alpha_k < \omega^n$, and $f : \gamma \longrightarrow \{0, \ldots, k\}$.* If $\forall i \leq k \quad \alpha_i = ot(f^{-1}[i])$ holds, then $\gamma < \omega^{\mathrm{n}}$.

Lemma 24. *Let k be some non-null integer, $(\mathbb{N}^k, \lesssim)$ be a well-ordering such that for every k-tuples $(a_0, \ldots, a_{k-1}), (b_0, \ldots, b_{k-1}) \in \mathbb{N}^k$ the following holds:*

$$(a_0, \ldots, a_{k-1}) \lesssim (b_0, \ldots, b_{k-1}) \Longrightarrow \begin{cases} \forall i < k \quad a_i \leq b_i \\ \text{or} \\ \exists i, j < k \quad \textit{such that } a_i < b_i \textit{ and } a_j > b_j. \end{cases}$$

Then, the order type of $(\mathbb{N}^k, \lesssim)$ is at most ω^k.

Lemma 25. *We let k be any non-null integer, $B \in \mathbf{BC(k)}$, $A \subseteq X^\omega$ be any initializable set, and δ any countable ordinal.*

$$B \leq_w A \bullet \delta \quad \Longrightarrow \quad B \leq_w A \bullet \alpha \text{ for some } \alpha < \omega^{k+1}.$$

An immediate consequence is that $B \equiv_w A \bullet \delta$ holds only for ordinals $\delta < \omega^{k+1}$.

Proof. First notice that for every $B \subseteq X^\omega$, and every $u \in X^*$, if $B \in \mathbf{BC(k)}$ holds, then $u^{-1}B \in \mathbf{BC(k)}$ holds too.
Towards a contradiction, we assume that $A \bullet \alpha <_w B \leq_w A \bullet \delta$ holds for all $\alpha < \omega^{k+1}$. We let $\mathcal{B}$ be a k-blind counter automaton that recognizes B. By Lemma 18, for each successor ordinal $\alpha < \omega^{k+1}$ there exists some $u_\alpha \in X^*$ such that $u_\alpha^{-1}B \equiv_w A \bullet \alpha$ or $u_\alpha^{-1}B \equiv_w (A \bullet \alpha)^{\complement}$. For each such u_α, we form $(q_\alpha, c_{\alpha,0}, c_{\alpha,1}, \ldots, c_{\alpha,k-1})$ where q_α denotes the control state that $\mathcal{B}$ is in after having read u_α, and $c_{\alpha,i}$ the height of its counter number i (any $i < k$).
Now there exists necessarily some control state q such that the order type of the set $S = \{\alpha < \omega^{k+1} \mid \alpha \text{ successor and } q_\alpha = q\}$ is ω^{k+1}. By Lemma 24 there exist $\alpha, \alpha' \in S$ such that $\alpha' < \alpha$ holds together with $c_{\alpha,i} \leq c'_{\alpha,i}$ (any $i < k$). (Without loss of generality, we may even assume that $\omega \leq \alpha' < \alpha$ holds.) Let us denote $\mathcal{B}_{\alpha'}$ the k-blind counter automaton $\mathcal{B}$ that starts in state $(q_{\alpha'}, c_{\alpha',0}, c_{\alpha',1}, \ldots, c_{\alpha',k-1})$, and $\mathcal{B}_\alpha$ the one that starts in state $(q_\alpha, c_{\alpha,0}, c_{\alpha,1}, \ldots, c_{\alpha,k-1})$. Notice that since $c_{\alpha,i} \leq c'_{\alpha,i}$ holds for all $i < k$, $\mathcal{B}_{\alpha'}$ performs exactly the same as $\mathcal{B}_\alpha$ except when the latter crashes for it tries to decrease a counter that is already empty. But it is then not difficult to see that given the above assumption – that $\omega \leq \alpha' < \alpha$ holds – $u_\alpha^{-1}B \leq_w u_{\alpha'}^{-1}B$ holds which leads to either $A\bullet\alpha \leq_w A\bullet\alpha'$ or $(A\bullet\alpha)^{\complement} \leq_w A\bullet\alpha'$. In both cases, it contradicts $\alpha' < \alpha$. □

Notice that $\emptyset \bullet \omega_1^n$ being initializable, we have in particular the following result.

Lemma 26. *For k, n any integers, A any non-self dual ω-language in $\mathbf{BC(k)}$, and any non-zero countable ordinal α, A or $A^{\complement} \equiv_w (\emptyset \bullet \omega_1^n) \bullet \alpha \Longrightarrow \alpha < \omega^{k+1}$.*

In a similar way, we may now state the following lemma.

Lemma 27. *We let k be any non-null integer, $B \in \mathbf{BC(k)}$, $A \subseteq X^\omega$ be any initializable set, δ be any countable ordinal, and C be any set of the form*

$$C = A \bullet \omega_1^n \bullet \nu_n + \cdots + A \bullet \omega_1^{n-1} \bullet \nu_{n-1} + \cdots + A \bullet \omega_1 \bullet \nu_1$$

for any non-zero integer n*, and countable multiplicative coefficients* $\nu_n, \nu_{n-1}, \dots, \nu_1$ *with at least one of them being non-null. Then we have*

$$B \leq_w C + A \bullet \delta \implies B \leq_w C + A \bullet \alpha \text{ for some } \alpha < \omega^{k+1}.$$

Theorem 28. *Let* k *be any non-null integer,* $B \subseteq X^\omega$ *be non-self dual. If* $B \in \mathbf{BC(k)}$*, then either* B *or* $B^{\complement}$ *is Wadge equivalent to some*

$$\Omega(\alpha) = (\emptyset \bullet \omega_1^{n_j}) \bullet \delta_j + (\emptyset \bullet \omega_1^{n_{j-1}}) \bullet \delta_{j-1} + \dots + (\emptyset \bullet \omega_1^{n_0}) \bullet \delta_0.$$

where $j \in \mathbb{N}$*,* $n_j > n_{j-1} > \ldots > n_0$ *and* $\omega^{k+1} > \delta_j, \delta_{j-1}, \dots, \delta_0 > 0$*.*

Proof. This is an almost immediate consequence of Lemmas 25 and 27. □

This settles the case of the non-self dual ω-languages in $\mathbf{BC(k)}$. For the self-dual ones, it is enough to notice the easy following:

1. Given any $A \subseteq X^\omega$, if $A \in \mathbf{BC(k)}$ is self dual, then there exists two non-self dual sets $B, C \subseteq X^\omega$ such that both B and C belong to $\mathbf{BC(k)}$, $B \equiv_w C^{\complement}$, and $A \equiv_w X_0 B \cup X_1 C$, where $\{X_0, X_1\}$ is any partition of X in two non-empty sets.
2. If $A \subseteq X^\omega$ and $B \subseteq X^\omega$ are non-self dual, verify $A \equiv_w B^{\complement}$, and both belong to $\mathbf{BC(k)}$, then, given any partition of X in two non-empty sets $\{X_0, X_1\}$, $X_0 A \cup X_1 B$ is self-dual, and also belongs to $\mathbf{BC(k)}$.

If we set $d^\circ(A) = sup\{d^\circ(B) + 1 \mid B <_W A\}$(any $A \subseteq X^\omega$), then we obtain that there exists an ω-language $B \subseteq X^\omega$ recognized by some deterministic Petri net, such that $A \equiv_w B$ holds iff $d^\circ A$ is of the form $\alpha = \omega_1^n \cdot \delta_n + \dots + \omega_1^0 \cdot \delta_0$ for some $n \in \mathbb{N}$, and $\omega^\omega > \delta_n, \dots, \delta_0 \geq 0$. Finally, an easy computation provides $(\omega^\omega)^\omega = \omega^{\omega^2}$ as the height of the Wadge hierarchy of ω-languages recognized by deterministic Petri nets.

7 Conclusions

We provided a description of the extension of the Wagner hierarchy from automata to deterministic Petri Nets with Muller acceptance conditions. The results are rigorously the same if we replace Muller acceptance conditions with parity acceptance conditions. But with Büchi acceptance conditions instead, it becomes even simpler since the ω-languages are no more boolean combinations of $\mathbf{\Sigma}_2^0$-sets, but $\mathbf{\Pi}_2^0$-sets. So, the whole hierarchy comes down to the following:

Corollary 29. *For any* $A \subseteq X^\omega$*, there exists an* ω*-language* $B \subseteq X^\omega$ *recognized by some deterministic Petri net with* Büchi acceptance conditions*, such that* $A \equiv_w B$ *iff either* $d^\circ A = \omega_1$*, and* A *is* $\mathbf{\Pi}_2^0$*-complete, or* $d^\circ A < \omega^\omega$*.*

Deciding the degree of a given ω-language in $\mathbf{BC(k)}$, for $k \geq 2$, recognized by some deterministic Petri net – either with *Büchi* or *Muller* acceptance conditions, remains an open question. Notice that for $k = 1$ this decision problem has been shown to be decidable by the second author in [11].

Another rather interesting open direction of research is to go from deterministic to non-deterministic Petri nets. It is clear that this step forward brings new Wadge classes – for instance there exist ω-languages recognized by non-deterministic Petri nets with *Büchi acceptance conditions* that are not $\mathbf{\Delta}^0_3$ [19] – but the description of this whole hierarchy still requires more investigations.

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